

Representation Functions for Jordanian Quantum Group $SL_h(2)$ and Jacobi Polynomials

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Abstract

The explicit expressions of the representation functions (D -functions) for Jordanian quantum group $SL_h(2)$ are obtained by combination of tensor operator technique and Drinfeld twist. It is shown that the D -functions can be expressed in terms of Jacobi polynomials as the undeformed D -functions can. Some of the important properties of the D -functions for $SL_h(2)$ such as Wigner's product law, recurrence relations, RTT type relations are also presented.

I. Introduction

It is known that quantum deformation of Lie group $GL(2)$ with central quantum determinant is classified into two types [1] : the standard deformation $GL_q(2)$ [2] and the Jordanian deformation $GL_h(2)$ [3, 4, 5]. The representation theory of $GL_q(2)$ has been studied extensively and we know that its contents are quite rich (See, for instance, Refs. [6, 7]). On the other hand, the representation theory of $GL_h(2)$ has not been developed yet. There are some works studying differential geometry on quantum h -plane and on $SL_h(2)$ itself [8]. However, the representation functions for $GL_h(2)$, the most basic ingredient of representation theories, has not been known. Recently Chakrabarti and Quesne [9] showed that the representation functions for two-parametric extension of $GL_h(2)$ [5, 10] can be obtained from the standard deformed ones via a contraction method and gave explicit form of representation functions for some low dimensional cases. In Ref.[11], the present author shows that the Jordanian deformation of symplecton for $sl(2)$ gives a natural basis for a representation of $SL_h(2)$ and he also gives another basis in terms of quantum h -plane.

The purpose of the present paper is to obtain explicit formulae for $SL_h(2)$ representation functions using the tensor operator technique and to investigate their properties. Representation functions are also called Wigner's D -functions in physicist's terminology. We use both terms and concentrate ourselves to the finite dimensional highest weight irreducible representations of $SL_h(2)$ throughout the present paper. In order to make a comparison between D -functions for $SL_q(2)$ and $SL_h(2)$, let us recall some known properties of D -functions for $SL_q(2)$ [12] : (a) Wigner's product law [13], (b) recurrence relations [13, 14], (c) orthogonality (d) RTT type relations [14], (e) D -functions can be written in terms of the little q -Jacobi polynomials [15], (f) generating function [16]. We will show, in this paper, that many of them have counterparts in the representation theory of $SL_h(2)$. Only exception is the generating function, it is not presented in this paper. Of course it does not mean that the generating function for the D -functions of $SL_h(2)$ does not exist.

The plan of this paper is as follows: we present the definitions of $SL_h(2)$ and its dual quantum algebra $\mathcal{U}_h(sl(2))$ in the next section. In §III, before deriving the explicit formulae for the representation functions, we discuss general features of them which are valid for any kind of deformation of $SL(2)$ under the assumption that the representation theory of the dual quantum algebra has a one-to-one correspondence with the undeformed

$sl(2)$. Then we shall write down the recurrence relations for $SL_h(2)$ D -functions. §IV is a brief review of the D -functions for Lie group $SL(2)$ (and $GL(2)$). We emphasize that the D -functions for $GL(2)$ form, in a certain boson realization, irreducible tensor operators of the Lie algebra $gl(2) \oplus gl(2)$. In §V, a tensor operator technique is used to obtain the boson realization of the generators of the Jordanian quantum group $GL_h(2)$, then it is generalized to obtain the D -functions for $GL_h(2)$. We shall apply the same technique to show that the D -functions for $SL_h(2)$ can be expressed in terms of Jacobi polynomials. This method will be applied to obtain a boson realization for two-parametric extension of the Jordanian deformation of $GL(2)$ in §VI. §VII is concluding remarks.

II. $SL_h(2)$ and its Dual

The Jordanian quantum group $GL_h(2)$ is generated by four elements x, y, u and v subject to the relations [3, 4, 5]

$$\begin{aligned} [v, x] &= hv^2, & [u, x] &= h(D - x^2), \\ [v, y] &= hv^2, & [u, y] &= h(D - y^2), \\ [x, y] &= h(xv - yv), & [v, u] &= h(xv + vy), \end{aligned} \quad (2.1)$$

where $D = xy - uv - h xv$ is the quantum determinant generating the center of $GL_h(2)$. This is a Hopf algebra and Hopf algebra mappings have a similar form as $GL_q(2)$. However, explicit form of the mappings is not necessary in the following discussion. By setting $D = 1$, we obtain $SL_h(2)$ from $GL_h(2)$.

The quantum algebra dual to $GL_h(2)$ is denoted by $\mathcal{U}_h(gl(2))$, and defined by the same commutation relations as the Lie algebra $gl(2)$

$$[J_0, J_{\pm}] = \pm 2J_{\pm}, \quad [J_+, J_-] = J_0, \quad [Z, \bullet] = 0. \quad (2.2)$$

However, their Hopf algebra mappings are modified via twisting [17] by the invertible element $\mathcal{F} \in \mathcal{U}_h(gl(2))^{\otimes 2}$ [18]

$$\mathcal{F} = \exp\left(-\frac{1}{2}J_0 \otimes \sigma\right), \quad \sigma = -\ln(1 - 2hJ_+). \quad (2.3)$$

The coproduct Δ , counit ϵ and antipode S for $\mathcal{U}_h(gl(2))$ are obtained from those for $gl(2)$ by

$$\Delta = \mathcal{F}\Delta_0\mathcal{F}^{-1}, \quad \epsilon = \epsilon_0, \quad S = \mu S_0 \mu^{-1}, \quad (2.4)$$

where the mappings with suffix 0 stand for the Hopf algebra mappings for $gl(2)$. The elements μ and μ^{-1} are defined, using the product m for $gl(2)$, by

$$\mu = m(id \otimes S_0)(\mathcal{F}), \quad \mu^{-1} = m(S_0 \otimes id)(\mathcal{F}^{-1}). \quad (2.5)$$

The twist element \mathcal{F} is not depend on the central element Z so that the Hopf algebra mappings for Z remain undeformed. Therefore the Jordanian quantum algebra obtained by the twist element (2.3) has the decomposition $\mathcal{U}_h(gl(2)) = \mathcal{U}_h(sl(2)) \oplus u(1)$. The Jordanian quantum algebra $\mathcal{U}_h(gl(2))$ is a triangular Hopf algebra whose universal R -matrix is given by $\mathcal{R} = \mathcal{F}_{12}\mathcal{F}^{-1}$.

It is obvious, from the commutation relation (2.2), that $\mathcal{U}_h(gl(2))$ and $gl(2)$ have the same finite dimensional highest weight irreducible representations. Furthermore we can easily see that tensor product of two irreducible representations (irreps) is completely reducible and decomposed into irreps in the same way as $gl(2)$, since the Clebsch-Gordan coefficients (CGC) for $\mathcal{U}_h(gl(2))$ are product of the ones for $gl(2)$ and matrix elements of the twist element \mathcal{F} . For the $\mathcal{U}_h(sl(2))$ sector, this is carried out in Ref.[11]. The CGC for $\mathcal{U}_h(sl(2))$ in another basis are discussed in Ref.[20]

Let Δ, ϵ be the coproduct and counit for $GL_h(2)$, respectively. We use the same notations for the Hopf algebra mappings of both $GL_h(2)$ and $\mathcal{U}_h(gl(2))$, however, this may not cause serious confusion. A vector space (representation space) V is called right $GL_h(2)$ comodule, if there exist a map $\rho : V \rightarrow V \otimes GL_h(2)$ such that the following relations are satisfied

$$(\rho \otimes id) \circ \rho = (id_V \otimes \Delta) \circ \rho, \quad (id_V \otimes \epsilon) \circ \rho = id_V, \quad (2.6)$$

where id_V stands for the identity map in V . The left comodule is defined in the similar manner. Using the bases $\{e_i \mid i = 1, 2, \dots, n\}$ of V , the map ρ is written as

$$\rho(e_i) = \sum_j e_j \otimes \mathcal{D}_{ji}, \quad (2.7)$$

it follows that the relation (2.6) are rewritten as

$$\Delta(\mathcal{D}_{ij}) = \sum_k \mathcal{D}_{ik} \otimes \mathcal{D}_{kj}, \quad \epsilon(\mathcal{D}_{ij}) = \delta_{ij}. \quad (2.8)$$

We call $\mathcal{D}_{ij} \in GL_h(2)$ satisfying (2.7) and (2.8) the D -function for $GL_h(2)$.

III. Properties of D -functions

A. Wigner's Product Law and RTT Type Relations

Before deriving the explicit formulae for $SL_h(2)$ D -functions, one can discuss some important properties of D -functions such as Wigner's product law, recurrence relations, RTT type relations and so on, using the definition of universal T -matrix [21, 22]. The explicit expression of the universal T -matrix is not necessary. The universal T -matrix for the standard deformation of $GL(2)$ is given in Ref.[22], while it is not known for the Jordanian deformation of $GL(2)$.

The discussion in this subsection is quite general. We shall present it so as to be applicable to any kind of deformation of $SL(2)$ (standard, Jordanian, two-parametric extension, anything else if any). Then we will write down the results explicitly for the Jordanian deformation of $SL(2)$ in the next subsection. It will also be seen that the discussion is easily extended to other groups.

Let \mathcal{G} and \mathfrak{g} be deformation of Lie group $SL(2)$ and Lie algebra $sl(2)$, respectively. The duality between \mathcal{G} and \mathfrak{g} are expressed, by choosing suitable bases, in terms of the universal T -matrix [22]. Let x^α and X_α be elements of a basis of \mathcal{G} and \mathfrak{g} , respectively. They are chosen as follows: the product is given by

$$x^\alpha x^\beta = \sum_\gamma h_\gamma^{\alpha,\beta} x^\gamma, \quad X_\alpha X_\beta = \sum_\gamma f_{\alpha,\beta}^\gamma X_\gamma, \quad (3.1)$$

the coproduct is given by

$$\Delta(x^\alpha) = \sum_{\beta,\gamma} f_{\beta,\gamma}^\alpha x^\beta \otimes x^\gamma, \quad \Delta(X_\alpha) = \sum_{\beta,\gamma} h_\alpha^{\beta,\gamma} X_\beta \otimes X_\gamma. \quad (3.2)$$

Then the universal T -matrix \mathcal{T} is defined by

$$\mathcal{T} = \sum_\alpha x^\alpha \otimes X_\alpha. \quad (3.3)$$

We assume that the deformed algebra \mathfrak{g} has the same finite dimensional highest weight irreps as $sl(2)$, that is, (1) each irrep is classified by the spin j and a irrep basis $|jm\rangle$ is specified by j and the magnetic quantum number m , (2) tensor product of irreps j_1 and j_2 is completely reducible

$$j_1 \otimes j_2 = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2|.$$

We further assume that vectors $|jm\rangle$ are complete and orthonormal. Then the D -functions for \mathcal{G} is obtained by

$$\mathcal{D}_{m',m}^j = \langle jm' | \mathcal{T} | jm \rangle = \sum_{\alpha} x^{\alpha} \langle jm' | X_{\alpha} | jm \rangle. \quad (3.4)$$

For the standard two-parametric deformation of $GL(2)$, the RHS of (3.4) was computed and it was shown that (3.4) coincided with the D -functions obtained by another method [23]. In our case, we show that the D -functions (3.4) satisfy (2.8) by making use of the relations (3.1) and (3.2). The coproduct of $\mathcal{D}_{m',m}^j$ is computed as

$$\begin{aligned} \Delta(\mathcal{D}_{m',m}^j) &= \sum_{\alpha} \Delta(x^{\alpha}) \langle jm' | X_{\alpha} | jm \rangle = \sum_{\beta,\gamma} x^{\beta} \otimes x^{\gamma} \langle jm' | X_{\beta} X_{\gamma} | jm \rangle \\ &= \sum_{\beta,\gamma,k} x^{\beta} \otimes x^{\gamma} \langle jm' | X_{\beta} | jk \rangle \langle jk | X_{\gamma} | jm \rangle = \sum_k \mathcal{D}_{m',k}^j \otimes \mathcal{D}_{k,m}^j. \end{aligned}$$

To compute the counit for $\mathcal{D}_{m',m}^j$, we use the identity obtained from the definition of counit

$$\sum_{\beta,\gamma} f_{\beta,\gamma}^{\alpha} \epsilon(x^{\beta}) x^{\gamma} = x^{\alpha}. \quad (3.5)$$

Using this relation, the universal T -matrix is rewritten as

$$\begin{aligned} \mathcal{T} &= \sum_{\alpha} x^{\alpha} \otimes X_{\alpha} = \sum_{\beta,\gamma} f_{\beta,\gamma}^{\alpha} \epsilon(x^{\beta}) x^{\gamma} \otimes X_{\alpha} = \sum_{\beta} \epsilon(x^{\beta}) x^{\gamma} \otimes X_{\beta} X_{\gamma} \\ &= \left(\sum_{\beta} \epsilon(x^{\beta}) \otimes X_{\beta} \right) \mathcal{T}. \end{aligned}$$

It follows that

$$\left(\sum_{\beta} \epsilon(x^{\beta}) \otimes X_{\beta} \right) = (\epsilon \otimes id)(\mathcal{T}) = 1. \quad (3.6)$$

Therefore the counit for D -functions is

$$\epsilon(\mathcal{D}_{m',m}^j) = \langle jm' | (\epsilon \otimes id)(\mathcal{T}) | jm \rangle = \langle jm' | jm \rangle = \delta_{m',m}.$$

We first show that the D -functions (3.4) satisfy the analogous relations to Wigner's product law. Let us denote the CGC for \mathbf{g} by $\Omega_{m_1, m_2, m}^{j_1, j_2, j}$, i.e.,

$$|(j_1 j_2) jm\rangle = \sum_{m_1, m_2} \Omega_{m_1, m_2, m}^{j_1, j_2, j} |j_1 m_1\rangle \otimes |j_2 m_2\rangle. \quad (3.7)$$

We write the inverse of the above relation as follows:

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{j, m} \mathcal{U}_{m_1, m_2, m}^{j_1, j_2, j} |(j_1 j_2) jm\rangle. \quad (3.8)$$

Then an analogue of Wigner's product law reads

Theorem 3.1 *The D-functions for \mathcal{G} satisfy the relation*

$$\delta_{j,j'} \mathcal{D}_{m',m}^j = \sum_{k_1,k_2,m_1,m_2} \mathcal{U}_{k_1,k_2,m'}^{j_1,j_2,j'} \Omega_{m_1,m_2,m}^{j_1,j_2,j} \mathcal{D}_{k_1,m_1}^{j_1} \mathcal{D}_{k_2,m_2}^{j_2}. \quad (3.9)$$

Proof : Because of the relations (3.1) and (3.2), one can show that

$$(id \otimes \Delta)(\mathcal{T}) = \sum_{\alpha,\beta} x^\alpha x^\beta \otimes X_\alpha \otimes X_\beta, \quad (\Delta \otimes id)(\mathcal{T}) = \sum_{\alpha,\beta} x^\alpha \otimes x^\beta \otimes X_\alpha X_\beta. \quad (3.10)$$

It follows that

$$(id \otimes \Delta)(\mathcal{T}) |(j_1 j_2) j m\rangle = \sum_{\alpha,\beta,m_1,m_2} \Omega_{m_1,m_2,m}^{j_1,j_2,j} x^\alpha x^\beta \otimes X_\alpha |j_1 m_1\rangle \otimes X_\beta |j_2 m_2\rangle. \quad (3.11)$$

The LHS of (3.11) is rewritten as

$$\sum_{m'} \mathcal{D}_{m',m}^j \otimes |(j_1 j_2) j m'\rangle = \sum_{m',k_1,k_2} \Omega_{k_1,k_2,m'}^{j_1,j_2,j} \mathcal{D}_{m',m}^j \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle.$$

The RHS of (3.11) is rewritten as

$$\begin{aligned} & \sum \Omega_{m_1,m_2,m}^{j_1,j_2,j} \langle j_1 k_1 | X_\alpha | j_1 m_1 \rangle \langle j_2 k_2 | X_\beta | j_2 m_2 \rangle x^\alpha x^\beta \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle \\ &= \sum \Omega_{m_1,m_2,m}^{j_1,j_2,j} \mathcal{D}_{k_1,m_1}^{j_1} \mathcal{D}_{k_2,m_2}^{j_2} \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle. \end{aligned}$$

Thus we obtain

$$\sum_{m'} \Omega_{k_1,k_2,m'}^{j_1,j_2,j} \mathcal{D}_{m',m}^j = \sum_{m_1,m_2} \Omega_{m_1,m_2,m}^{j_1,j_2,j} \mathcal{D}_{k_1,m_1}^{j_1} \mathcal{D}_{k_2,m_2}^{j_2}. \quad (3.12)$$

Using the orthogonality of $\Omega_{m_1,m_2,m}^{j_1,j_2,j}$ and $\mathcal{U}_{m_1,m_2,m}^{j_1,j_2,j}$, the theorem is proved. \square

Corollary 3.2 *The D-functions also satisfy the following relations*

$$\sum_{m'} \Omega_{k_1,k_2,m'}^{j_1,j_2,j} \mathcal{D}_{m',m}^j = \sum_{m_1,m_2} \Omega_{m_1,m_2,m}^{j_1,j_2,j} \mathcal{D}_{k_1,m_1}^{j_1} \mathcal{D}_{k_2,m_2}^{j_2}, \quad (3.13)$$

$$\sum_m \mathcal{U}_{m_1,m_2,m}^{j_1,j_2,j} \mathcal{D}_{m',m}^j = \sum_{k_1,k_2} \mathcal{U}_{k_1,k_2,m'}^{j_1,j_2,j} \mathcal{D}_{k_1,m_1}^{j_1} \mathcal{D}_{k_2,m_2}^{j_2}, \quad (3.14)$$

$$\mathcal{D}_{k_1,m_1}^{j_1} \mathcal{D}_{k_2,m_2}^{j_2} = \sum_{j,m,m'} \mathcal{U}_{m_1,m_2,m}^{j_1,j_2,j} \Omega_{k_1,k_2,m'}^{j_1,j_2,j} \mathcal{D}_{m',m}^j. \quad (3.15)$$

Proof : (3.13) has already been obtained in the proof of Theorem 3.1, see (3.12). Others can be obtained from (3.13) by the orthogonality of Ω and \mathcal{U} . \square

For $\mathcal{G} = SL_q(2)$ and $\mathfrak{g} = \mathcal{U}_q(sl(2))$, the CGC Ω, \mathcal{U} are given by the q -analogue of the CGC of $sl(2)$: $\Omega_{m_1, m_2, m}^{j_1, j_2, j} = \mathcal{U}_{m_1, m_2, m}^{j_1, j_2, j} = {}_q C_{m_1, m_2, m}^{j_1, j_2, j}$. From the relations (3.9) and (3.13) - (3.15), the recurrence relations and the orthogonality of $SL_q(2)$ D -functions are obtained [6, 13, 14].

Next we show that the D -functions (3.4) satisfy the RTT type relation.

Theorem 3.3 *The D -functions for \mathcal{G} satisfy*

$$\sum_{s_1, s_2} (R^{j_1, j_2})_{m_1, m_2}^{s_1, s_2} \mathcal{D}_{s_1, k_1}^{j_1} \mathcal{D}_{s_2, k_2}^{j_2} = \sum_{s_1, s_2} \mathcal{D}_{m_2, s_2}^{j_2} \mathcal{D}_{m_1, s_1}^{j_1} (R^{j_1, j_2})_{s_1, s_2}^{k_1, k_2}, \quad (3.16)$$

where $(R^{j_1, j_2})_{m_1, m_2}^{s_1, s_2}$ are the matrix elements of the universal R -matrix for \mathfrak{g}

$$(R^{j_1, j_2})_{m_1, m_2}^{s_1, s_2} = \langle j_1 m_1 | \otimes \langle j_2 m_2 | \mathcal{R} | j_1 s_1 \rangle \otimes | j_2 s_2 \rangle .$$

Remark : For $j_1 = j_2 = 1/2$, the matrix elements for \mathcal{R} are evaluated in the fundamental representation of \mathfrak{g} . Therefore, the relation (3.16) is reduced to the defining relation of \mathcal{G} in FRT-formalism [2]. This implies that $\mathcal{D}_{m', m}^{\frac{1}{2}}$ are generators of \mathcal{G} .

Proof : The relation (3.16) can be proved by evaluating matrix elements of the RTT type relation for the universal T -matrix [24]. We define

$$\mathcal{T}_1 = \sum x^\alpha \otimes X_\alpha \otimes 1, \quad \mathcal{T}_2 = \sum x^\alpha \otimes 1 \otimes X_\alpha,$$

then

$$\begin{aligned} \mathcal{T}_1 \mathcal{T}_2 &= \sum_{\alpha, \beta} x^\alpha x^\beta \otimes X_\alpha \otimes X_\beta = \sum_{\alpha} x^\alpha \otimes \Delta(X_\alpha), \\ \mathcal{T}_2 \mathcal{T}_1 &= \sum_{\alpha, \beta} x^\beta x^\alpha \otimes X_\alpha \otimes X_\beta = \sum_{\alpha} x^\alpha \otimes \Delta'(X_\alpha), \end{aligned}$$

where Δ' stands for the opposite coproduct. It follows that

$$\mathcal{T}_2 \mathcal{T}_1 = \sum_{\alpha} x^\alpha \otimes \mathcal{R} \Delta(X_\alpha) \mathcal{R}^{-1},$$

thus we obtain

$$(1 \otimes \mathcal{R}) \mathcal{T}_1 \mathcal{T}_2 = \mathcal{T}_2 \mathcal{T}_1 (1 \otimes \mathcal{R}).$$

Evaluating the matrix elements on $1 \otimes |j_1 k_1\rangle \otimes |j_2 k_2\rangle$, the theorem is proved. \square

For $\mathcal{G} = SL_q(2)$, the relation (3.16) was proved by Nomura [14]. However, Theorem 3.3 shows that the relation (3.16) holds for any kind of deformation of $SL(2)$. In Ref.[14], the D -functions for $SL_q(2)$ are interpreted as the wave functions of quantum symmetric tops in noncommutative space.

B. Recurrence Relations and Orthogonality-like Relations

In this subsection, the recurrence relations and the orthogonality-like relations of the $SL_h(2)$ D -functions are derived as a consequence of the theorems in the previous subsection. It is known that the CGC for $\mathcal{U}_h(sl(2))$ are given in terms of the CGC for $sl(2)$ and the matrix elements of the twist element \mathcal{F}

$$\Omega_{m_1, m_2, m}^{j_1, j_2, j} = \sum_{s_1, s_2} C_{s_1, s_2, m}^{j_1, j_2, j} (F^{j_1, j_2})_{m_1, m_2}^{s_1, s_2}, \quad (3.17)$$

where $C_{s_1, s_2, m}^{j_1, j_2, j}$ is the CGC for $sl(2)$ and $(F^{j_1, j_2})_{m_1, m_2}^{s_1, s_2}$ is given by

$$(F^{j_1, j_2})_{m_1, m_2}^{s_1, s_2} = \langle j_1, m_1 | \otimes \langle j_2, m_2 | \mathcal{F} | j_1, s_1 \rangle \otimes | j_2, s_2 \rangle.$$

The explicit formula for $(F^{j_1, j_2})_{m_1, m_2}^{s_1, s_2}$ and the next relation are found in Ref.[11].

$$(F^{j_1, j_2})_{-m_1, -m_2}^{-s_1, -s_2} = ((F^{-1})^{j_1, j_2})_{s_1, s_2}^{m_1, m_2}. \quad (3.18)$$

The CGC for $\mathcal{U}_h(sl(2))$ satisfy the orthogonality relations [20] because of

$$\mathcal{U}_{m_1, m_2, m}^{j_1, j_2, j} = (-1)^{j_1 + j_2 - j} \Omega_{-m_1, -m_2, -m}^{j_1, j_2, j} = \sum_{s_1, s_2} C_{s_1, s_2, m}^{j_1, j_2, j} ((F^{-1})^{j_1, j_2})_{s_1, s_2}^{m_1, m_2}. \quad (3.19)$$

The relation(3.18) and the well known property of the $sl(2)$ CGC are used in the last equality.

Note that we have known the following fact because of the remark to Theorem 3.3.

Proposition 3.4 $\mathcal{D}_{m', m}^{\frac{1}{2}}$ are the generators of $SL_h(2)$

$$\begin{pmatrix} \mathcal{D}_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} & \mathcal{D}_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} \\ \mathcal{D}_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} & \mathcal{D}_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} x & u \\ v & y \end{pmatrix}. \quad (3.20)$$

Let us consider the case that j_1 is arbitrary and $j_2 = 1/2$ in order to derive the recurrence relations for $SL_h(2)$ D -functions. In this case, the F -coefficients have a simple form

$$\begin{aligned} (F^{j_1, \frac{1}{2}})_{k_1, k_2}^{m_1, \frac{1}{2}} &= \delta_{k_1, m_1} \delta_{k_2, \frac{1}{2}}, \\ (F^{j_1, \frac{1}{2}})_{k_1, k_2}^{m_1, -\frac{1}{2}} &= \delta_{k_1, m_1} (\delta_{k_2, -\frac{1}{2}} - 2m_1 h \delta_{k_2, \frac{1}{2}}), \\ ((F^{-1})^{j_1, \frac{1}{2}})_{m_1, \frac{1}{2}}^{n_1, n_2} &= \delta_{m_1, n_1} (\delta_{n_2, \frac{1}{2}} + 2m_1 h \delta_{n_2, -\frac{1}{2}}), \\ ((F^{-1})^{j_1, \frac{1}{2}})_{m_1, -\frac{1}{2}}^{n_1, n_2} &= \delta_{m_1, n_1} \delta_{n_2, -\frac{1}{2}}. \end{aligned}$$

One can use Winger's product law, expressed in the form of (3.13) and (3.14), to derive the recurrence relations for $\mathcal{D}_{m',m}^j$ which are reduced to the known recurrence relations of the $SL(2)$ D -functions in the limit of $h = 0$.

Proposition 3.5 *The $SL_h(2)$ D -functions satisfy the following recurrence relations*

$$\begin{aligned}
\text{(i)} \quad & \sqrt{j+k}\mathcal{D}_{k,m}^j - (2k-1)h\sqrt{j-k+1}\mathcal{D}_{k-1,m}^j \\
& = \sqrt{j+m}\mathcal{D}_{k-\frac{1}{2},m-\frac{1}{2}}^{j-\frac{1}{2}}x + \sqrt{j-m}\mathcal{D}_{k-\frac{1}{2},m+\frac{1}{2}}^{j-\frac{1}{2}}(u - (2m+1)hx), \\
\text{(ii)} \quad & \sqrt{j-k}\mathcal{D}_{k,m}^j = \sqrt{j+m}\mathcal{D}_{k+\frac{1}{2},m-\frac{1}{2}}^{j-\frac{1}{2}}v + \sqrt{j-m}\mathcal{D}_{k+\frac{1}{2},m+\frac{1}{2}}^{j-\frac{1}{2}}(y - (2m+1)hv), \\
\text{(iii)} \quad & \sqrt{j+n}\mathcal{D}_{m,n}^j = \sqrt{j+m}\mathcal{D}_{m-\frac{1}{2},n-\frac{1}{2}}^{j-\frac{1}{2}}(x + (2m-1)hv) + \sqrt{j-m}\mathcal{D}_{m+\frac{1}{2},n-\frac{1}{2}}^{j-\frac{1}{2}}v, \\
\text{(iv)} \quad & \sqrt{j-n}\mathcal{D}_{m,n}^j + \sqrt{j+n}(2n+1)h\mathcal{D}_{m,n+1}^j \\
& = \sqrt{j+m}\mathcal{D}_{m-\frac{1}{2},n+\frac{1}{2}}^{j-\frac{1}{2}}(u + (2m-1)hy) + \sqrt{j-m}\mathcal{D}_{m+\frac{1}{2},n+\frac{1}{2}}^{j-\frac{1}{2}}y, \\
\text{(v)} \quad & \sqrt{j-k+1}\mathcal{D}_{k,m}^j + (2k-1)h\sqrt{j+k}\mathcal{D}_{k-1,m}^j \\
& = \sqrt{j-m+1}\mathcal{D}_{k-\frac{1}{2},m-\frac{1}{2}}^{j+\frac{1}{2}}x - \sqrt{j+m+1}\mathcal{D}_{k-\frac{1}{2},m+\frac{1}{2}}^{j+\frac{1}{2}}(u - (2m+1)hx), \\
\text{(vi)} \quad & \sqrt{j+k+1}\mathcal{D}_{k,m}^j = -\sqrt{j-m+1}\mathcal{D}_{k+\frac{1}{2},m-\frac{1}{2}}^{j+\frac{1}{2}}v \\
& + \sqrt{j+m+1}\mathcal{D}_{k+\frac{1}{2},m+\frac{1}{2}}^{j+\frac{1}{2}}(y - (2m+1)hv), \\
\text{(vii)} \quad & \sqrt{j-n+1}\mathcal{D}_{m,n}^j = \sqrt{j-m+1}\mathcal{D}_{m-\frac{1}{2},n-\frac{1}{2}}^{j+\frac{1}{2}}(x + (2m-1)hv) \\
& - \sqrt{j+m+1}\mathcal{D}_{m+\frac{1}{2},n-\frac{1}{2}}^{j+\frac{1}{2}}v, \\
\text{(viii)} \quad & \sqrt{j+n+1}\mathcal{D}_{m,n}^j - \sqrt{j-n}(2n+1)h\mathcal{D}_{m,n+1}^j \\
& = -\sqrt{j-m+1}\mathcal{D}_{m-\frac{1}{2},n+\frac{1}{2}}^{j+\frac{1}{2}}(u + (2m-1)hy) + \sqrt{j+m+1}\mathcal{D}_{m+\frac{1}{2},n+\frac{1}{2}}^{j+\frac{1}{2}}y.
\end{aligned}$$

Proof : Put $j_2 = 1/2$, $j = j_1 + 1/2$ in the relation (3.13), then

$$\begin{aligned}
& \sqrt{j_1+k_1+1}\mathcal{D}_{k_1+\frac{1}{2},m}^{j_1+\frac{1}{2}} + \sqrt{j_1-k_1+1}(\delta_{k_2,-\frac{1}{2}} - 2k_1h\delta_{k_2,\frac{1}{2}})\mathcal{D}_{k_1-\frac{1}{2},m}^{j_1+\frac{1}{2}} \\
& = \sqrt{j_1+m+\frac{1}{2}}\mathcal{D}_{k_1,m-\frac{1}{2}}^{j_1}\mathcal{D}_{k_2,\frac{1}{2}}^{\frac{1}{2}} + \sqrt{j_1-m+\frac{1}{2}}\mathcal{D}_{k_1,m+\frac{1}{2}}^{j_1}(\mathcal{D}_{k_2,-\frac{1}{2}}^{\frac{1}{2}} - (2m+1)h\mathcal{D}_{k_2,\frac{1}{2}}^{\frac{1}{2}}).
\end{aligned}$$

Replacing $j_1 + \frac{1}{2}$ and $k_1 + \frac{1}{2}$ with j and k , respectively, we obtain

$$\begin{aligned}
& \sqrt{j+k}\delta_{k_2,\frac{1}{2}}\mathcal{D}_{k,m}^j + \sqrt{j-k+1}(\delta_{k_2,-\frac{1}{2}} - (2k-1)h\delta_{k_2,\frac{1}{2}})\mathcal{D}_{k-1,m}^j \\
& = \sqrt{j+m}\mathcal{D}_{k-\frac{1}{2},m-\frac{1}{2}}^{j-\frac{1}{2}}\mathcal{D}_{k_2,\frac{1}{2}}^{\frac{1}{2}} + \sqrt{j-m}\mathcal{D}_{k-\frac{1}{2},m+\frac{1}{2}}^{j-\frac{1}{2}}(\mathcal{D}_{k_2,-\frac{1}{2}}^{\frac{1}{2}} - (2m+1)h\mathcal{D}_{k_2,\frac{1}{2}}^{\frac{1}{2}}).
\end{aligned}$$

The recurrence relations (i) and (ii) are obtained by putting $k_2 = 1/2$ and $k_2 = -1/2$, respectively.

We repeat the similar computation for the relation (3.14). We put $j_2 = 1/2$, $j = j_1 + 1/2$ in (3.14), then rearrange some variables. We obtain

$$\begin{aligned} & \sqrt{j+n}\{\delta_{n_2, \frac{1}{2}} + (2n-1)h\delta_{n_2, -\frac{1}{2}}\}\mathcal{D}_{m,n}^j + \sqrt{j-n+1}\delta_{n_2, -\frac{1}{2}}\mathcal{D}_{m,n-1}^j \\ = & \sqrt{j+m}\mathcal{D}_{m-\frac{1}{2}, n-\frac{1}{2}}^{j-\frac{1}{2}}\{\mathcal{D}_{\frac{1}{2}, n_2}^{\frac{1}{2}} + (2m-1)h\mathcal{D}_{-\frac{1}{2}, n_2}^{\frac{1}{2}}\} + \sqrt{j-m}\mathcal{D}_{m+\frac{1}{2}, n-\frac{1}{2}}^{j-\frac{1}{2}}\mathcal{D}_{-\frac{1}{2}, n_2}^{\frac{1}{2}}. \end{aligned}$$

The recurrence relations (iii) and (iv) correspond to the cases of $n_2 = 1/2$ and $n_2 = -1/2$, respectively.

The recurrence relations (v) - (viii) correspond to $j_2 = 1/2$, $j = j_1 - 1/2$. In this case, the relation (3.13) yields after rearrangement of variables

$$\begin{aligned} & \sqrt{j-k+1}\delta_{k_2, \frac{1}{2}}\mathcal{D}_{k,m}^j - \sqrt{j+k}(\delta_{k_2, -\frac{1}{2}} - (2k-1)h\delta_{k_2, \frac{1}{2}})\mathcal{D}_{k-1,m}^j \\ = & \sqrt{j-m+1}\mathcal{D}_{k-\frac{1}{2}, m-\frac{1}{2}}^{j+\frac{1}{2}}\mathcal{D}_{k_2, \frac{1}{2}}^{\frac{1}{2}} - \sqrt{j+m+1}\mathcal{D}_{k-\frac{1}{2}, m+\frac{1}{2}}^{j+\frac{1}{2}}(\mathcal{D}_{k_2, -\frac{1}{2}}^{\frac{1}{2}} - (2m+1)h\mathcal{D}_{k_2, \frac{1}{2}}^{\frac{1}{2}}). \end{aligned}$$

Putting $k_2 = 1/2$ and $-1/2$, we obtain the relations (v) and (vi), respectively. The relation (3.14) yields

$$\begin{aligned} & \sqrt{j-n+1}(\delta_{n_2, \frac{1}{2}} + (2n-1)h\delta_{n_2, -\frac{1}{2}})\mathcal{D}_{m,n}^j - \sqrt{j+n}\delta_{n_2, -\frac{1}{2}}\mathcal{D}_{m,n-1}^j \\ = & \sqrt{j-m+1}\mathcal{D}_{m-\frac{1}{2}, n-\frac{1}{2}}^{j+\frac{1}{2}}(\mathcal{D}_{\frac{1}{2}, n_2}^{\frac{1}{2}} + (2m-1)h\mathcal{D}_{-\frac{1}{2}, n_2}^{\frac{1}{2}}) - \sqrt{j+m+1}\mathcal{D}_{m+\frac{1}{2}, n-\frac{1}{2}}^{j+\frac{1}{2}}\mathcal{D}_{-\frac{1}{2}, n_2}^{\frac{1}{2}}. \end{aligned}$$

The recurrence relations (vii) and (viii) are obtained as the cases of $n_2 = 1/2$ and $n_2 = -1/2$, respectively. \square

It is possible to obtain the explicit form of D -functions for some special cases such as $\mathcal{D}_{m',j}^j$, $\mathcal{D}_{j,m}^j$, by solving these recurrence relations. However, it seems to be difficult to derive formulae for $\mathcal{D}_{m',m}^j$ for any values of j , m' and m . We will solve this problem by using the tensor operator approach in §V.

The orthogonality-like relations for $\mathcal{D}_{m',m}^j$ can be obtained from the relations (3.13) and (3.14).

Proposition 3.6 *The D -functions for $SL_h(2)$ $\mathcal{D}_{m',m}^j$ satisfy the orthogonality-like relations which are reduced to the orthogonality relations of $SL(2)$ D -functions in the limit of $h = 0$.*

$$\sum_{m_1, m_2} (-1)^{k_1-m_1} (F^{j,j})_{m_1, m_2}^{m_1, -m_1} \mathcal{D}_{k_1, m_1}^j \mathcal{D}_{k_2, m_2}^j = (F^{j,j})_{k_1, k_2}^{k_1, -k_1}, \quad (3.21)$$

$$\sum_{k_1, k_2} (-1)^{m_1-k_1} ((F^{-1})^{j,j})_{k_1, -k_1}^{k_1, k_2} \mathcal{D}_{k_1, m_1}^j \mathcal{D}_{k_2, m_2}^j = ((F^{-1})^{j,j})_{m_1, -m_1}^{m_1, m_2}. \quad (3.22)$$

Proof : Consider the cases of $j = 0, j_1 = j_2$ in the relations (3.13) and (3.14). Writing $j_1 = j_2 = j$, they yield

$$\begin{aligned} \sum_{m_1, m_2} \Omega_{m_1, m_2, 0}^{j, j, 0} \mathcal{D}_{k_1, m_1}^j \mathcal{D}_{k_2, m_2}^j &= \Omega_{k_1, k_2, 0}^{j, j, 0}, \\ \sum_{k_1, k_2} \mathcal{U}_{k_1, k_2, 0}^{j, j, 0} \mathcal{D}_{k_1, m_1}^j \mathcal{D}_{k_2, m_2}^j &= \mathcal{U}_{m_1, m_2, 0}^{j, j, 0}. \end{aligned}$$

The CGC are given by

$$\Omega_{m_1, m_2, 0}^{j, j, 0} = \sum_s C_{s, -s, 0}^{j, j, 0} (F^{j, j})_{m_1, m_2}^{s, -s}, \quad \mathcal{U}_{m_1, m_2, 0}^{j, j, 0} = \sum_s C_{s, -s, 0}^{j, j, 0} ((F^{-1})^{j, j})_{s, -s}^{m_1, m_2},$$

and

$$(F^{j, j})_{m_1, m_2}^{s, -s} = \delta_{s, m_1} \langle j m_2 | e^{-s\sigma} | j - s \rangle, \quad ((F^{-1})^{j, j})_{s, -s}^{m_1, m_2} = \delta_{s, m_1} \langle j - s | e^{m_1 \sigma} | j m_2 \rangle.$$

Then the proof of Proposition 3.6 is straightforward. \square

IV. Review of $SL(2)$ Representation Functions

This section is devoted to a review of the D -functions for Lie group $SL(2)$. Especially, we focus on tensor operator properties and the relationship to Jacobi polynomials. We write the D -functions for $SL(2)$ in terms of boson operators for the viewpoint of tensor operators.

Let $a_i^j, \bar{a}_i^j, i, j \in \{1, 2\}$ be four copies of a boson operator commuting one another, *i.e.*,

$$[\bar{a}_i^j, a_k^\ell] = \delta_{i, k} \delta^{j, \ell}, \quad [a_i^j, a_k^\ell] = [\bar{a}_i^j, \bar{a}_k^\ell] = 0. \quad (4.1)$$

It is known that the Lie algebra $gl(2) \oplus gl(2)$ is realized by these boson operators. The left (lower) generators are defined by

$$E_{ij} = a_i^1 \bar{a}_j^1 + a_i^2 \bar{a}_j^2, \quad (4.2)$$

the right (upper) generators are defined by

$$E^{ij} = a_1^i \bar{a}_1^j + a_2^i \bar{a}_2^j. \quad (4.3)$$

Then both left and right generators satisfy the $gl(2)$ commutation relations and furthermore $[E_{ij}, E^{k, \ell}] = 0$. Each $gl(2)$ has decomposition $gl(2) = sl(2) \oplus u(1)$. The left and right $sl(2)$ are generated by

$$J_+ = E_{21}, \quad J_- = E_{12}, \quad J_0 = E_{22} - E_{11}, \quad (4.4)$$

and

$$K_+ = E^{12}, \quad K_- = E^{21}, \quad K_0 = E^{11} - E^{22}, \quad (4.5)$$

respectively, and $u(1)$ sectors by $Z_L = -E_{11} - E_{22}$ and $Z_R = E^{11} + E^{22}$. This choice of generators may be different from the usual one (see for example Ref.[6] §4.4). However this is a suitable choice for twisting discussed in the next section. Note also that, in this realization, $Z_L = -Z_R$. Therefore, strictly speaking, this realization is not the direct sum of two copies of $gl(2)$.

The D -functions for Lie group $GL(2)$ can be given in terms of a_i^j

$$\mathcal{D}_{m',m}^{(0)j} = \{(j+m')!(j-m')!(j+m)!(j-m)!\}^{1/2} \sum_{K,L,M,N} \frac{(a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N}{K!L!M!N!}, \quad (4.6)$$

where the sum over K, L, M and N runs nonnegative integers provided that

$$\begin{aligned} K + L &= j + m, & M + N &= j - m, \\ K + M &= j + m', & L + N &= j - m'. \end{aligned} \quad (4.7)$$

We obtain $SL(2)$ D -functions by imposing $a_1^1 a_2^2 - a_2^1 a_1^2 = 1$.

It is not difficult to see that D -functions (4.6) form the irreducible tensor operators for both left and right $gl(2)$, *i.e.*,

$$\begin{aligned} [J_\pm, \mathcal{D}_{m',m}^{(0)j}] &= \sqrt{(j \pm m')(j \mp m' + 1)} \mathcal{D}_{m' \mp 1, m}^{(0)j}, \\ [J_0, \mathcal{D}_{m',m}^{(0)j}] &= -2m' \mathcal{D}_{m',m}^{(0)j}, \quad [Z_L, \mathcal{D}_{m',m}^{(0)j}] = -2j \mathcal{D}_{m',m}^{(0)j}, \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} [K_\pm, \mathcal{D}_{m',m}^{(0)j}] &= \sqrt{(j \mp m)(j \pm m + 1)} \mathcal{D}_{m', m \pm 1}^{(0)j}, \\ [K_0, \mathcal{D}_{m',m}^{(0)j}] &= 2m \mathcal{D}_{m',m}^{(0)j}, \quad [Z_R, \mathcal{D}_{m',m}^{(0)j}] = 2j \mathcal{D}_{m',m}^{(0)j}. \end{aligned} \quad (4.9)$$

It is well known that the D -functions for $SL(2)$ can be expressed in terms of the Jacobi polynomials. The Jacobi polynomials are defined by

$$P_n^{(\alpha, \beta)}(z) = \sum_{r=0}^n \frac{(-n)_r (\alpha + \beta + n + 1)_r}{(1)_r (\alpha + 1)_r} z^r, \quad (4.10)$$

where $(\alpha)_r$ stands for the sifted factorial

$$(\alpha)_r = \alpha(\alpha + 1) \cdots (\alpha + r - 1).$$

For the case of $SL(2)$, we have the relation $a_1^1 a_2^2 = 1 + a_2^1 a_1^2$. Using this, the D -functions are expressed for $m' + m \geq 0$, $m' \geq m$

$$\mathcal{D}_{m',m}^{(0)j} = \left\{ \binom{j+m'}{m'-m} \binom{j-m}{m'-m} \right\}^{1/2} (a_1^1)^{m'+m} (a_1^2)^{m'-m} P_{j-m'}^{(m'-m, m'+m)}(z), \quad (4.11)$$

where $z \equiv -a_2^1 a_1^2$. We have the similar relations for other cases.

V. Representation Functions for $SL_h(2)$

A. Explicit Formulae for D -Functions

We saw, in the previous section, that the D -functions for $GL(2)$ form the irreducible tensor operators of both left and right $gl(2)$. This fact leads us to the expectation that the D -functions for $GL_h(2)$ also form the irreducible tensor operators of left and right $\mathcal{U}_h(gl(2))$. It is known that the tensor operators for $\mathcal{U}_h(gl(2))$ can be obtained from the ones for $gl(2)$ by twisting [11, 25]. Therefore, we may obtain the D -functions for $GL_h(2)$ from the one for $GL(2)$ by twisting twice. The irreducible tensor operators for $\mathcal{U}_h(gl(2))$ are defined by replacing the comutator on the left hand side of (4.8) and (4.9) with the adjoint action. Let \mathbf{t} be a any tensor operator for $\mathcal{U}_h(gl(2))$ and $X \in \mathcal{U}_h(gl(2))$, then the adjoint action of X on \mathbf{t} is defined by [26]

$$\text{ad}X(\mathbf{t}) = m(id \otimes S)(\Delta(X)(\mathbf{t} \otimes 1)). \quad (5.1)$$

The tensor operators \mathbf{t} for $\mathcal{U}_h(gl(2))$ and the tensor operators $\mathbf{t}^{(0)}$ for $gl(2)$ are related via the twist element \mathcal{F} by the relation [25] (see also Ref. [11])

$$\mathbf{t} = m(id \otimes S)(\mathcal{F}(\mathbf{t}^{(0)} \otimes 1)\mathcal{F}^{-1}). \quad (5.2)$$

Note that $gl(2)$ and $\mathcal{U}_h(gl(2))$ have the same commutation relations so that the realization (4.2) and (4.3) is the realization of $\mathcal{U}_h(gl(2))$ as well. We consider the tensor operators under this realization of $\mathcal{U}_h(gl(2))$.

Let us first consider the simplest case : $j = 1/2$. What we obtain in this case from (4.6), (4.8) and (4.9) is that the pairs (a_1^1, a_2^1) , (a_1^2, a_2^2) are spinors of the left $gl(2)$ and the pairs (a_1^1, a_1^2) , (a_2^1, a_2^2) are spinors of the right $gl(2)$. Namely, each boson operator a_i^j is a component of spinor for both left and right $gl(2)$. This fact tells us that, by twisting

via the elements

$$\mathcal{F}_L = \exp\left(-\frac{1}{2}J_0 \otimes \sigma_L\right), \quad \mathcal{F}_R = \exp\left(-\frac{1}{2}K_0 \otimes \sigma_R\right). \quad (5.3)$$

with $\sigma_L = -\ln(1 - 2hJ_+)$, $\sigma_R = -\ln(1 - 2hK_+)$, we obtain a element of spinor for both left and right $\mathcal{U}_h(sl(2))$. To this end, it is convenient to rewrite the relation (5.2) into different form. Let us write the twist element and its inverse as

$$\mathcal{F} = \sum_a f^a \otimes f_a, \quad \mathcal{F}^{-1} = \sum_a g^a \otimes g_a,$$

then

$$\mu = \sum_a f^a S_0(f_a), \quad \mu^{-1} = \sum_a S_0(g^a) g_a.$$

Noting the identity

$$\sum g^b \mu S_0(g_b) = \sum g^b f^a S_0(g_b f_a) = m(id \otimes S_0)(\mathcal{F}^{-1} \mathcal{F}) = 1,$$

the relation (5.2) yields

$$\mathbf{t} = \sum f^a \mathbf{t}^{(0)} g^b S(f_a g_b) = \sum f^a \mathbf{t}^{(0)} \mu S_0(f_a g_b) \mu^{-1} = \sum f^a \mathbf{t}^{(0)} S_0(f_a) \mu^{-1}. \quad (5.4)$$

From (5.4), the twisting by \mathcal{F}_L reads

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)^n J_0^n a_i^j S_0(\sigma_L) \mu^{-1} = a_i^j \sum_{k=0}^{\infty} \frac{(-1)^{ik}}{k!} \left(-\frac{1}{2}\right)^k S(\sigma_L)^k = a_i^j \exp\{(-1)^i \sigma_L/2\}.$$

We used the fact $S(\sigma_L) = -\sigma_L$ in the last equality. To twist the above obtained result by \mathcal{F}_R , we can repeat the similar computation. Then we have the doubly twisted boson operators

$$a_i^j \exp\{(-1)^i \sigma_L/2 + (-1)^{j+1} \sigma_R/2\}. \quad (5.5)$$

The commutation relations of the twisted boson operators (5.5) are obtained by straightforward computation and it shows that the twisted boson operators give a realization of generators of $GL_h(2)$.

Proposition 5.1 *Let*

$$\begin{aligned} x &= a_1^1 e^{(-\sigma_L + \sigma_R)/2}, & u &= a_1^2 e^{-(\sigma_L + \sigma_R)/2}, \\ v &= a_2^1 e^{(\sigma_L + \sigma_R)/2}, & y &= a_2^2 e^{(\sigma_L - \sigma_R)/2}, \end{aligned} \quad (5.6)$$

then, x , u , v and y satisfy the commutation relations of the generators of $GL_h(2)$ (2.1).

In this realization, the central element D is given

$$D \equiv xy - uv - h xv = a_1^1 a_2^2 - a_2^1 a_1^2. \quad (5.7)$$

Note that the central element D remains undeformed in this realization.

Proof : One can verify the commutation relations directly. We here give some useful commutation relations for the verification. The commutation relations between σ_L , σ_R and boson operators.

$$\begin{aligned} [\sigma_L, a_1^1] &= 2he^{\sigma_L} a_2^1, & [\sigma_L, a_1^2] &= 2he^{\sigma_L} a_2^2, \\ [\sigma_R, a_1^2] &= 2he^{\sigma_R} a_1^1, & [\sigma_R, a_2^2] &= 2he^{\sigma_R} a_2^1. \end{aligned}$$

These are easily verified by using the power series expansion of σ_L, σ_R : $\sigma_L = \sum_{n=1}^{\infty} \frac{(2hJ_+)^n}{n}$. These relations can be used to prove the following commutation relations which hold for any real k

$$\begin{aligned} [e^{k\sigma_L}, a_1^1] &= 2hke^{(k+1)\sigma_L} a_2^1, & [e^{k\sigma_L}, a_1^2] &= 2hke^{(k+1)\sigma_L} a_2^2, \\ [e^{k\sigma_R}, a_1^2] &= 2hke^{(k+1)\sigma_R} a_1^1, & [e^{k\sigma_R}, a_2^2] &= 2hke^{(k+1)\sigma_R} a_2^1. \end{aligned} \quad (5.8)$$

□

Next let us consider the twisting of $\mathcal{D}_{m',m}^{(0)j}$ for any values of j by the twist elements $\mathcal{F}_L, \mathcal{F}_R$. We denote the doubly twisted $\mathcal{D}_{m',m}^{(0)j}$ by $\mathcal{D}_{m',m}^j$, since it will be shown later that this $\mathcal{D}_{m',m}^j$ gives the D -functions for $GL_h(2)$. The computation is almost same as the case of spinors. What we need to compute is the twisting of $(a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N$ in the expression (4.6). The twisting by \mathcal{F}_L reads

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)^n J_0^n (a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N S_0^n(\sigma_L) \mu^{-1} \\ &= (a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{1}{2}\right)^k (-K + L - M + N)^k \mu S_0^k(\sigma_L) \mu^{-1} \\ &= (a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N \exp\{-(K - L + M - N)\sigma_L/2\}. \end{aligned}$$

Further twisting by \mathcal{F}_R gives

$$(a_1^1)^K (a_2^1)^L (a_1^2)^M (a_2^2)^N \exp\{-(K - L + M - N)\sigma_L/2 + (K + L - M - N)\sigma_R/2\}. \quad (5.9)$$

Because of the condition (4.7), we have $K - L + M - N = 2m'$ and $K + L - M - N = 2m$. Thus the exponential factor appeared in (5.9) is factored out the sum over K, L, M and N . Therefore we have proved the following proposition.

Proposition 5.2 *In the realization (4.2), (4.3), the irreducible tensor operators of both left and right $\mathcal{U}_h(gl(2))$ are given by*

$$\mathcal{D}_{m',m}^j = \mathcal{D}_{m',m}^{(0)j} e^{-m'\sigma_L + m\sigma_R}. \quad (5.10)$$

One can write $\mathcal{D}_{m',m}^j$ of Proposition 5.2 in terms of the generators of $GL_h(2)$ by making use of Proposition 5.1. For real A, B ,

$$\begin{aligned} (a_1^1)^K e^{(A\sigma_L + B\sigma_R)/2} &= (a_1^1)^{K-1} x e^{(A+1)\sigma_L/2 + (B-1)\sigma_R/2} \\ &= (a_1^1)^{K-1} e^{(A+1)\sigma_L/2 + (B-1)\sigma_R/2} \{e^{-(A+1)\sigma_L/2} x e^{(A+1)\sigma_L/2}\}. \end{aligned}$$

The expression $\{\dots\}$ in the last line can be calculated by using (5.8) and gives $x - h(A+1)v$. Thus we have obtained

$$\begin{aligned} (a_1^1)^K e^{(A\sigma_L + B\sigma_R)/2} &= e^{(A+K)\sigma_L/2 + (B-K)\sigma_R/2} \\ &\quad \times (x - h(A+K)v)(x - h(A+K-1)v) \cdots (x - h(A+1)v). \end{aligned} \quad (5.11)$$

Similar computation gives three other identities

$$\begin{aligned} (a_2^1)^L e^{(A\sigma_L + B\sigma_R)/2} &= e^{(A-L)\sigma_L/2 + (B-L)\sigma_R/2} v^L, \\ (a_1^2)^M e^{(A\sigma_L + B\sigma_R)/2} &= e^{(A+M)\sigma_L/2 + (B+M)\sigma_R/2} \\ &\quad \times (u - h(B+M)x - h(A+M)y + h^2(A+M)(B+M)v) \\ &\quad \times (u - h(B+M-1)x - h(A+M-1)y + h^2(A+M-1)(B+M-1)v) \\ &\quad \times \cdots \times (u - h(B+1)x - h(A+1)y + h^2(A+1)(B+1)v), \\ (a_2^2)^N e^{(A\sigma_L + B\sigma_R)/2} &= e^{(A-N)\sigma_L/2 + (B+N)\sigma_R/2} \\ &\quad \times (y - h(B+N)v)(y - h(B+N-1)v) \cdots (y - h(B+1)v). \end{aligned} \quad (5.12)$$

The boson operators a_i^j commute one another so that the order of a_i^j 's in $\mathcal{D}_{m',m}^{(0)j}$ is irrelevant. Therefore we can have some different expressions of $\mathcal{D}_{m',m}^j$ according to the choice of the order of boson operators. We here give two of them and shall show that they are the representation functions of $GL_h(2)$.

Proposition 5.3 *The D-functions for $GL_h(2)$ are given by*

$$\mathcal{D}_{m',m}^j = \{(j+m')!(j-m')!(j+m)!(j-m)!\}^{1/2} \sum_{K,L,M,L} \frac{X_K v^L U_{K,L,M} Y_{K,L,M,N}}{K!M!L!N!}, \quad (5.13)$$

where X_K , $U_{K,L,M}$ and $Y_{K,L,M,N}$ are defined by

$$\begin{aligned}
X_K &= x(x + hv) \cdots (x + h(K - 1)v), \\
U_{K,L,M} &= (u - h(K + L)x + h(K - L)y - h^2(K^2 - L^2)v) \\
&\quad \times (u - h(K + L - 1)x + h(K - L + 1)y - h^2(K^2 - (L - 1)^2)v) \\
&\quad \times \cdots \\
&\quad \times (u - h(K + L - M + 1)x + h(K - L + M - 1)y \\
&\quad \quad - h^2(K^2 - (L - M + 1)^2)v) \\
Y_{K,L,M,N} &= (y - h(K + L - M)v)(y - h(K + L - M - 1)v) \\
&\quad \times \cdots (y - h(K + L - M - N + 1)v).
\end{aligned}$$

The D -functions have another expression which is

$$\mathcal{D}_{m',m}^j = \{(j + m')!(j - m')!(j + m)!(j - m)!\}^{1/2} \sum_{K,L,M,N} \frac{U_M X_{K,M} Y_{K,M,N} v^L}{K!M!L!N!}, \quad (5.14)$$

where U_M , $X_{K,M}$, $Y_{K,M,N}$ are defined by

$$\begin{aligned}
U_M &= u(u + h(x + y) + h^2v) \cdots (u + h(M - 1)(x + y) + h^2(M - 1)^2v), \\
X_{K,M} &= (x + hMv)(x + h(M + 1)v) \cdots (x + h(K + M - 1)v), \\
Y_{K,M,N} &= (y - h(K - M)v)(y - h(K - M - 1)v) \cdots (y - h(K - M - N + 1)v)v^L.
\end{aligned}$$

The sum over K, L, M and N runs nonnegative integers under the condition (4.7).

Remark : We obtain the D -functions for $SL_h(2)$ by putting $D = xy - uv - h xv = 1$.

Proof : These expressions are obtained by using (5.11) and (5.12). The expression (5.13) corresponds to the boson ordering $(a_1^1)^k (a_2^1)^L (a_1^2)^M (a_2^2)^N$, while the expression (5.14) corresponds to $(a_1^2)^M (a_1^1)^K (a_2^2)^N (a_2^1)^L$.

To show that $\mathcal{D}_{m',m}^j$ are the representation functions of $GL_h(2)$, we must verify (2.8). It is obvious that $\mathcal{D}_{m',m}^j \in GL_h(2)$ and the counit of $\mathcal{D}_{m',m}^j$ is easily verified by using $\epsilon(x) = \epsilon(y) = 1$, $\epsilon(u) = \epsilon(v) = 0$. However, it seems to be difficult to verify the coproduct of $\mathcal{D}_{m',m}^j$ by straightforward computation. Instead of verifying the coproduct, we show that $\mathcal{D}_{m',m}^j$ satisfy the recurrence relations of Proposition 3.5. Note that the recurrence relations of Proposition 3.5 are for $SL_h(2)$. The Jordanian deformation of the Lie algebra

$gl(2)$ considered in this paper is the direct sum of the deformed $sl(2)$ and undeformed $u(1)$: $\mathcal{U}_h(gl(2)) = \mathcal{U}_h(sl(2)) \oplus u(1)$. This implies that the CGC for $\mathcal{U}_h(sl(2))$ also give the CGC for $\mathcal{U}_h(gl(2))$. Therefore the D -functions for $GL_h(2)$ also satisfy the recurrence relations of Proposition 3.5.

As an example, we show that the $\mathcal{D}_{m',m}^j$ give the solutions to the recurrence relation (ii). We substitute the expression (5.14) of the D -functions into the first term of the RHS of the relation (ii), then replace the dummy index L with $L - 1$. It follows that

$$\sqrt{j+m}\mathcal{D}_{k-\frac{1}{2},m-\frac{1}{2}}^{j-\frac{1}{2}}v = \{(j+m)!(j-m)!(j+k-1)!(j-k)!\}^{1/2} \sum_{K,L,M,N} L \frac{U_M X_{K,M} Y_{K,M,N} v^L}{K!M!L!N!},$$

where the indices K, L, M and N satisfy the condition

$$K + L = j + m, \quad M + N = j - m, \quad K + M = j + k - 1, \quad L + N = j - k + 1. \quad (5.15)$$

For the second term of the RHS of the relation (ii), we use the expression (5.13). Replacing the index N with $N - 1$, we obtain

$$\begin{aligned} & \sqrt{j-m}\mathcal{D}_{k-\frac{1}{2},m+\frac{1}{2}}^{j-\frac{1}{2}}(u - (2m+1)hv) \\ &= \{(j+m)!(j-m)!(j+k-1)!(j-k)!\}^{1/2} \sum_{K,L,M,N} N \frac{X_K v^L U_{K,L,M} Y_{K,L,M,N}}{K!M!L!N!}, \end{aligned}$$

where the indices K, L, M and N also satisfy the condition (5.15). Since the expressions (5.13) and (5.14) are the different expressions of the same D -functions, it holds that $U_M X_{K,M} Y_{K,M,N} v^L = X_K v^L U_{K,L,M} Y_{K,L,M,N}$. Therefore the RHS of (ii) reads

$$\begin{aligned} & \{(j+m)!(j-m)!(j+k-1)!(j-k)!\}^{1/2} \sum_{K,L,M,N} (L+N) \frac{X_K v^L U_{K,L,M} Y_{K,L,M,N}}{K!M!L!N!} \\ &= \sqrt{j-k+1}\mathcal{D}_{k-1,m}^j. \end{aligned}$$

The four-term recurrence relation (i) is reduced to a three-term relation, by eliminating $\mathcal{D}_{k-1,m}^j$ from (i) and (ii). This recurrence relation is easily solved by using the another expression of $\mathcal{D}_{k,m}^j$ corresponding to another ordering of boson operators. The suitable expressions for solving it are the ones obtained from the ordering $(a_1^2)^M (a_2^2)^N (a_2^1)^L (a_1^1)^K$ and $(a_1^1)^K (a_2^2)^N (a_2^1)^L (a_1^2)^M$. In this way, we can verify the $\mathcal{D}_{m',m}^j$ obtained in this Proposition solve all the recurrence relations given in Proposition 3.5. \square

Both expression of (5.13) and (5.14), of course, give the generators of $GL_h(2)$ for $j = 1/2$ which reflects Proposition 3.4. The D -functions for $j = 1$ reads

$$\mathcal{D}^1 = \begin{pmatrix} x^2 + h xv & \sqrt{2}(ux + huv) & u^2 + h(ux + uy + huv) \\ \sqrt{2}xv & D + 2uv & \sqrt{2}(uy + huv) \\ v^2 & \sqrt{2}yv & y^2 + hyv \end{pmatrix}. \quad (5.16)$$

For $SL_h(2)$, *i.e.* putting $D = 1$, this coincides with the one obtained by using h -symplecton or quantum h -plane [11]. Chakrabarti and Quesne obtained the \mathcal{D}^1 for two-parametric Jordanian deformation of $GL(2)$ in the coloured representation through a contraction technique to the D -functions for standard (q, λ) -deformation of $GL(2)$ [9]. To compare the present \mathcal{D}^1 with the one given in Ref.[9], put $\alpha = 0$, $z = 1$ in Eqs.(4.20) and (4.21) of Ref.[9]. Then we see that the D -functions for $j = 1$ of Ref. [9] are different form (5.16). This difference stems from the different choice of the basis of $\mathcal{U}_h(sl(2))$. In Ref.[9], the basis introduced by Ohn [27] is used, that is, the commutation relations of the generators of $\mathcal{U}_h(sl(2))$ are not same as those of $sl(2)$. While the basis of this paper satisfy the same commutation relations as $sl(2)$. This results the different CGC for the same algebra so that the recurrence relations for the D -functions have the different form. The CGC for Ohn's basis are found in Ref.[20]. Repeating the same procedure as §III.B, we obtain another form of recurrence relations. It may be easy to verify that the \mathcal{D}^1 of Ref.[9] solves these recurrence relations.

B. $SL_h(2)$ D -Functions and Jacobi Polynomials

The purpose of this subsection is to show that the D -functions for $SL_h(2)$ can be expressed in terms of Jacobi polynomials. To this end, we return to the boson realization of D -functions (Proposition 5.2) and use the fact that the D -functions for Lie group $SL(2)$ are written in terms of Jacobi polynomials. Recall the following two facts : (1) the central element D of $GL_h(2)$ is not deformed in the boson realization, Eq.(5.7), (2) Jacobi polynomials appeared in the D -functions for $SL(2)$ are power series in the variable $z = -a_2^1 a_1^2$. We write the D -functions $\mathcal{D}_{m',m}^{(0),j}$ for $SL(2)$ appeared in (5.10) in terms of Jacobi polynomials then use the easily proved relation $(a_2^1 a_1^2)^r = (uv)^r$ in order to replace the variable $z = -a_2^1 a_1^2$ with the h -deformed one $z = -uv$. Let us consider, as an example, the case of $m' + m \geq 0$, $m' \geq m$. The $\mathcal{D}_{m',m}^{(0),j}$ are given by (4.11). We rearrange the

order of a_1^1 , a_1^2 and $P_{j-m'}^{(m'-m, m'+m)}(z)$ to be $P_{j-m'}^{(m'-m, m'+m)}(z)(a_1^2)^{m'-m}(a_1^1)^{m'+m}$. Using the relations (5.11) and (5.12), we see that

$$\begin{aligned} & (a_1^2)^{m'-m}(a_1^1)^{m'+m}e^{-m'\sigma_L+m\sigma_R} \\ &= u(u+h(x+y)+h^2v)\cdots(u+h(m'-m-1)(x+y)+h^2(m'-m-1)^2v) \\ &\times (x+h(m'-m)v)(x+h(m'-m-1)v)\cdots(x+h(2m'-1)v). \end{aligned}$$

This completes the expression of D -functions in terms of Jacobi polynomials.

Repeating this process for other cases, we can prove the next proposition.

Proposition 5.4 *The D -functions for $SL_h(2)$ are written in terms of Jacobi polynomials as follows:*

(i) $m' + m \geq 0$, $m' \geq m$

$$\begin{aligned} \mathcal{D}_{m',m}^j &= N_+ P_{j-m'}^{(m'-m, m'+m)}(z) \\ &\times u(u+h(x+y)+h^2v)\cdots(u+h(m'-m-1)(x+y)+h^2(m'-m-1)^2v) \\ &\times (x+h(m'-m)v)(x+h(m'-m-1)v)\cdots(x+h(2m'-1)v). \end{aligned}$$

(ii) $m' + m \geq 0$, $m' \leq m$

$$\mathcal{D}_{m',m}^j = N_- P_{j-m}^{(-m'+m, m'+m)}(z)x(x+hv)\cdots(x+h(m'+m-1)v)v^{-m'+m}$$

(iii) $m' + m \leq 0$, $m' \geq m$

$$\begin{aligned} \mathcal{D}_{m',m}^j &= N_+ P_{j+m}^{(m'-m, -m'-m)}(z) \\ &\times u(u+h(x+y)+h^2v)\cdots(u+h(m'-m-1)(x+y)+h^2(m'-m-1)^2v) \\ &\times (y-h(m-m')v)(y-h(m-m'-1)v)\cdots(y-h(2m+1)v). \end{aligned}$$

(iv) $m' + m \leq 0$, $m' \leq m$

$$\begin{aligned} \mathcal{D}_{m',m}^j &= N_- P_{j+m'}^{(-m'+m, -m'-m)}(z) \\ &\times v^{-m'+m} (y-h(m-m')v)(y-h(m-m'-1)v)\cdots(y-h(2m+1)v). \end{aligned}$$

The variable z is defined by $z = -uv$ and the factors N_+ , N_- by

$$N_+ = \left\{ \begin{pmatrix} j+m' \\ m'-m \end{pmatrix} \begin{pmatrix} j-m \\ m'-m \end{pmatrix} \right\}^{1/2}, \quad N_- = \left\{ \begin{pmatrix} j-m' \\ m-m' \end{pmatrix} \begin{pmatrix} j+m \\ m-m' \end{pmatrix} \right\}^{1/2}.$$

Remark : The Jacobi polynomials are to the left of the generators of $SL_h(2)$. To move $P_n^{(\alpha, \beta)}(z)$ to the right, the relation

$$(uv)^r \exp(-m'\sigma_L + m\sigma_R) = \exp(-m'\sigma_L + m\sigma_R) \{uv - 2h(-m'yv + mxv) - 4h^2mm'v^2\}^r,$$

is used and we see that the Jacobi polynomials are changed to the power series in $\zeta_{m',m} = -(u + 2h(m'y - mx) - 4h^2mm')v$, but the rests of the formulae remain unchanged.

VI. Boson Realization of $GL_{h,g}(2)$

It is natural to generalize the results in the previous section to the two-parametric Jordanian deformation of $GL(2)$ [28], since the twist element which generates the two-parametric Jordanian quantum algebra $\mathcal{U}_{h,g}(gl(2))$ [29, 30] is known [31]. Unfortunately, the method in the previous sections leads us to quite complex calculation. As the first step to obtain the D -functions for two-parametric Jordanian quantum group $GL_{h,g}(2)$, we here give the boson realization of the generators of $GL_{h,g}(2)$.

The left and right twist elements are given by

$$\begin{aligned}\mathcal{F}_L &= \exp\left(\frac{g}{2h}\sigma_L \otimes Z_L\right) \exp\left(-\frac{1}{2}J_0 \otimes \sigma_L\right), \\ \mathcal{F}_R &= \exp\left(\frac{g}{2h}\sigma_R \otimes Z_R\right) \exp\left(-\frac{1}{2}K_0 \otimes \sigma_R\right),\end{aligned}$$

respectively. We can see that the $GL_{h,g}(2)$ is reduced to $GL_h(2)$ when $g = 0$. Repeating the same procedure as (5.5), we obtain the twisted boson operators. We can rewrite the twisted boson operator in terms of the generators $GL_h(2)$. The next proposition can be regarded as a realization of $GL_{h,g}(2)$ by generators of $GL_h(2)$ and Z_L, Z_R as well.

Proposition 6.1 *Let*

$$\begin{aligned}a &= x - gvZ_L, & b &= u - gxZ_R - gyZ_L + g^2vZ_LZ_R, \\ c &= v, & d &= y - gvZ_R,\end{aligned}\tag{6.1}$$

where x, u, v and y are given by (5.6). Then a, b, c and d satisfy the commutation relation of $GL_{h,g}(2)$.

Remark : In this realization, the quantum determinant $D' = ad - bc - (h + g)ac$ for $GL_{h,g}(2)$ and D for $GL_h(2)$ coincide : $D' = D = a_1^1a_2^2 - a_2^1a_1^2$.

Proof : It requires a lengthy calculation, however, the proof is straightforward. The following commutation relations [28] are verified.

$$\begin{aligned}
[a, b] &= -(h+g)(D' - a^2), & [a, c] &= -(h-g)c^2, \\
[a, d] &= (h+g)ac - (h-g)dc, & [b, c] &= -(h+g)ac - (h-g)cd, \\
[b, d] &= (h-g)(D' - d^2), & [c, d] &= (h+g)c^2.
\end{aligned} \tag{6.2}$$

□

VII. Concluding Remarks

In this paper, the explicit formulae of the D -functions for $SL_h(2)$ (and $GL_h(2)$) have been obtained by using the tensor operator technique. We used the fact that the D -functions for Lie group $GL(2)$ form irreducible tensor operators of $gl(2) \oplus gl(2)$ in the realization (4.2), (4.3). This kind of tensor operators are called double irreducible tensor operators in the literature. The D -functions for $GL_h(2)$ were obtained via the construction of double irreducible tensor operators for $\mathcal{U}_h(gl(2)) \oplus \mathcal{U}_h(gl(2))$. Other examples of double irreducible tensor operators were considered for q -deformation [32, 33] and for Jordanian deformation [34]. Quesne constructed the $GL_h(n) \times GL_{h'}(m)$ covariant bosonic and fermionic algebra which form the double irreducible tensor operators of $\mathcal{U}_h(gl(n)) \oplus \mathcal{U}_{h'}(gl(m))$ using the contraction method [34]. This may suggest, in the case of $n = m = 2$ and $h = h'$, that the bosonic algebra of Quesne has a close relation to $\mathcal{D}_{m', m}^{\frac{1}{2}}$, *i.e.*, the generators of $GL_h(2)$.

We also showed that the D -functions for $SL_h(2)$ can be expressed in terms of Jacobi polynomials. Contrary to the q -deformed case where the little q -Jacobi polynomials appear in the D -functions for $SU_q(2)$, the ordinary Jacobi polynomials are associated with the D -functions for $SL_h(2)$. It seems to be a general feature of Jordanian deformation that the ordinary orthogonal polynomials are associated with the representations. It is known that the ordinary Gauss hypergeometric functions are associated with h -symplecton [11], while the q -hypergeometric functions are associated with the q -deformation of symplecton.

The extension of the results of this paper to the Jordanian deformation of $SL(n)$ may be possible, since the explicit expressions for the twist element are known for the Lie algebra $sl(n)$ [35].

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